# The Application of the Hardy Cross Method of Moment Distribution

By H. A. WILLIAMS,[1] STANFORD UNIVERSITY P. O., CALIF.

This paper presents the basic principles of the Hardy Cross method of analyzing continuous frames by distributing fixed-end moments and illustrates the application of the method to various types of structures, including an elevator spar with supports on a line, the same spar with deflected supports, and an airplane fuselage truss with loads between panel points.
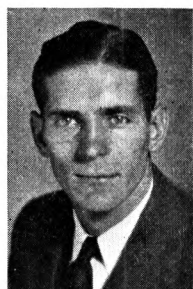
FOR years, structural engineers have been searching for a simple method of determining stresses in statically indeterminate structures. The Maxwell-Mohr, least-work, and slope-deflection methods have been employed to a large extent, but the laborious computations involved, together with the tendency for small errors to accumulate, have discouraged their use.

In May, 1930, Prof. Hardy Cross, of the University of Illinois, presented in the Proceedings of the American Society of Civil Engineers a paper entitled, "Analysis of Continuous Frames by Distributing Fixed End Moments."[2] The method proper was one of successive approximation involving only the simplest arithmetic and eliminating the use of simultaneous equations. The widespread discussion which followed would seem ample proof of the profession's interest in this method. Professor Cross purposely limited his paper to definitions and to a brief illustration and discussion of the application of the method to one type of frame. He also suggested other possible applications, many of which have been covered in subsequent discussion.

Due to the brevity of the original paper, it seems desirable that a more detailed presentation be made in which emphasis should be placed on the correlation between moment distribution and the physical action of a structure. The writer has attempted to accomplish this by simple illustrations, while explaining the basic principles.

Only a few practical applications to aeronautical problems have been given in this paper. But it is felt that familiarity with the basic principles should enable the airplane designer to apply the method to his special problems without difficulty.

The author is very much indebted to Prof. A. S. Niles for the criticisms and suggestions that he has given. Numerous features of the paper have been instigated by him.

DEFINITIONS

In order to clarify the immediate discussion of the Hardy Cross method, the sign conventions and four terms will be defined as follows:

(1) *Fixed-end moment* is the moment which would exist at the ends of a loaded member if those ends were held rigidly against rotation. This is in accord with the ordinary usage for beams ending in heavy walls.[3]

(2) *Unbalanced moment* is numerically equal to the algebraic sum of all moments at a joint. The joint is balanced when the sum of the moments equals zero.

(3) *Stiffness* as herein used "is the moment at one end of a member (which is on unyielding supports at both ends) necessary to produce unit rotation of that end when the other end is fixed."[4] For a straight beam of constant section, the stiffness is proportional to the moment of inertia divided by the span length— i.e., the usual $K = I/L$ used in the slope deflection method. Of course, the relative rather than the actual values of K can be used in computations.

(4) *Carry-over factors* result from the fact that if a beam simply supported at one end and fixed at the other is acted on by a bending moment at the simply supported end, a certain moment is induced at or "carried over" to the fixed end. The ratio of the moment at the fixed end to the bending moment at the simply supported end is called the "carry-over factor." It is $+ 1/2$ for straight beams of constant section.[5]

(5) The *sign conventions* used in this paper are as follows: A clockwise moment couple acting on the end of a member is positive; a counter-clockwise moment couple is negative. If, at a joint, the tangent to the elastic curve of a member rotates through a clockwise angle, the rotation is positive. Hence, a positive moment acting on the end of a member causes a positive rotation.[6]

BASIC PRINCIPLES

It will be recalled that, in the analysis of statically indeterminate structures, various temporary expedients are resorted to in order to obtain the moments and stresses in the members. Redundant members are assumed to be cut or a fixed or partially restrained joint is taken as hinged, and the resulting statically determinate structure is analyzed for this condition. Then certain external forces or moments are assumed to come into

[3] See Appendix B for fixed-end moments for beams with various types of loading.

[4] Quoted from Hardy Cross paper. See Trans. Am.Soc.C.E., vol. 96, 1932, p. 2.

[5] See footnote 10.

[6] The sign conventions adopted here are the same as those used by Messrs. Wilson, Richart, and Weiss in "Statically Indeterminate Structures," Bulletin 108, University of Illinois, Engineering Experiment Station. They also are used by Sutherland and Bowman in their text, "Structural Theory." Note that these sign conventions bear no relation to the signs used in ordinary design, and hence are not the same as those used in the original paper by Hardy Cross. See Appendix A for more extensive definitions of sign conventions.

action and draw the cut member together or return the hinged joint to the restrained condition, and the same statically determinate structure is separately analyzed for the effects of the latter forces or moments. The principle of superposition[7] is assumed to apply, and results of the separate analyses are added algebraically to determine the final stresses. The procedure involves the application to the structure of certain arbitrary restraining forces or moments. These forces or moments are later removed, leaving the frame acted on by the original loads. The arbitrary restraints can be removed and re-applied any number of times as long as in the end the true force system prevails.

Consider, for illustration, a simple beam uniformly loaded. By fixing or restraining the ends, which are normally free to rotate, we can prevent the beam from sagging as far as it ordinarily would when similarly loaded and resting on knife-edged supports. If we now release the restraint at one end only, further deflection will take place; and the subsequent removal of the restraint at the other end will allow the beam to sag to its final position. In this case only two steps are involved. However, by alternately releasing and fixing the ends, the beam could have been lowered to its final position in as many stages as was desired. Each stage would have resulted in certain changes in the end moments of the beam. By the principle of superposition, the effects of all of these separate changes could be added algebraically to determine the final moments. Obviously, this summation would be zero for the end moments of this particular structure.

ILLUSTRATION, SIMPLE BEAM OF ONE SPAN

Suppose, in order to make the foregoing example more specific, we follow through in detail the steps suggested, using Figs. 1 to 9 to help in visualizing the physical action which takes place.[8]

(1) Fig. 1: We have the simple beam AB resting on unyielding knife-edged supports. The uniformly distributed load has not yet been applied.

(2) Fig. 2: It is assumed that before the load is applied, the ends are fixed by casting a heavy imaginary wall around the beam at each end.

(3) Fig. 3: The load is assumed to be applied. By well-known methods,[9] the end moments are found to be $-WL/12$ at A and $+WL/12$ at B as shown. By definition, these are "fixed-end moments."

(4) Fig. 4: The restraint at A is removed. As a result: (a) The moment of $-WL/12$ at A is entirely released. This effect is the same as though a new external moment of $+WL/12$ had been exerted at A. Hence, adding $+WL/12$ to the previous $-WL/12$ gives zero, the new moment at A. (b) It is obvious from the figure that the action of the external moment ($+WL/12$) at A caused additional strain or moment to be set up at B, the far end of the beam. This is the "carry-over" moment from A to B. It is equal to $+\frac{1}{2} \times +WL/12 = +WL/24$.[10] The net moment at B is now $+WL/8$, as would be expected.[11]

(5) Fig. 5: Joint A[12] is assumed fixed in its new position.

(6) Fig. 6: Joint B is released; i.e., an external moment of $-WL/8$ is applied, thus canceling the previously existing moment of $+WL/8$. The beam sags still further and sets up a strain at the fixed end A. That is, a moment of $+\frac{1}{2} \times -WL/8 = -WL/16$ is carried over to A and added to the existing moment, which is zero.

(7) Fig. 7: Joint B is again fixed.

(8) Fig. 8: Joint A is released, leaving zero moment at A and a carry-over moment of $+WL/32$ at B.

(9) Fig. 9: If we now decide arbitrarily to release B without

— Fig. 1 —

— Fig. 2 —

— Fig. 3 —

— Fig. 4 —

— Fig. 5 —

— Fig. 6 —

— Fig. 7 —

— Fig. 8 —

— Fig. 9 —

---

[7] The principle of superposition states that the total effect of a number of loads on a structure is the same as the sum of the effects of each load when separately applied. It is only approximately true for combined bending and compression.

[8] Since we are concerned only with moments at this time, no shears or support reactions will be shown in Figs. 1 to 9.

[9] That is, by theorem of three moments, moment areas, or slope deflection. Moments for fixed-ended beams of one span can also be found in any handbook.

[10] The ratio $+\frac{1}{2}$ is the carry-over factor for a straight beam of constant section. See Appendix B for derivation.

[11] The foregoing moments for a beam fixed at one end and hinged at the other can be checked from any handbook.
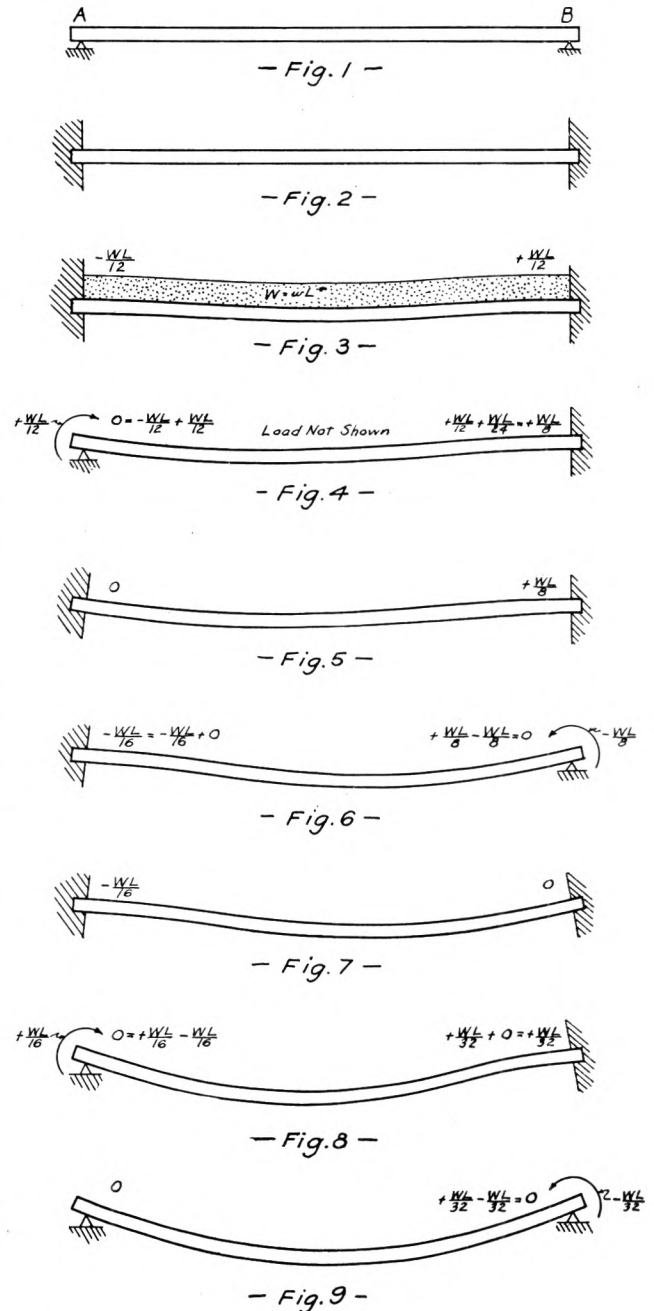
[12] In this paper, the term "joint" will be used not only to describe the point of intersection of the axes of two or more members as in a truss, but also to refer to the intersection of the axis of a beam and the center line of a support.

first fixing A, as was previously done, we can do so, and the beam reaches its final sag with zero moment at each end.

Inspection shows that each step brought us closer to the true condition for the loaded beam. Each time a moment was added due to a change in position of the beam, the added value was smaller than before, since the carry-over factor was $+1/2$. The principle of superposition was applied at each step in that the new moment was added to the existing moment before proceeding to the next step.

The foregoing example is of course extremely simple. Since the beam is of one span only and the ends are both hinged, it can be allowed to deflect to its final position at any time by simply releasing all restraining moments and allowing both end moments to become zero. However, this can be done only because both ends actually are hinged and we know that the final end moments must equal zero. In even a slightly more complicated structure, such as a continuous beam on three supports, the situation is different. Certain moments will exist at the interior ends of the spans when the structure reaches its final state of equilibrium. The magnitudes of these moments are unknown as a rule. Hence, the moment over the middle support resulting from releasing both end joints at once will not equal the actual final moment except by coincidence.

ILLUSTRATION, CONTINUOUS BEAM

Let us now follow through the analysis, in the same general manner as before, of the continuous beam shown in Fig. 10. The cross-section is constant throughout, and all spans are of equal length. The supports are assumed as unyielding. The only external force is the load P on the end of the overhang. Moments are designated in the usual fashion. M_{ab} is the moment in beam AB at end A; M_{ba} is the moment in beam AB at end B.

Assume that, before the load P is applied, all joints are fixed against rotation. This fixity can be thought of as being effected by casting a heavy wall around the beam at each support. Under these circumstances, the only immediate result of applying the load P is that a fixed-end moment ($M_{ao} = +1000$ ft-lb) is set up in the cantilever. All other moments equal zero. From this point, the procedure is as follows:

(1) Balance joint A: Suppose joint A is now released; i.e., the wall is assumed replaced by a knife-edged support. If $\Sigma M = M_{ao} + M_{ab} = 0$, the joint would be in equilibrium and no rotation would take place. However, $\Sigma M = +1000 + 0 = +1000$, which means that an unbalanced moment of $+1000$ exists at the joint for an instant after the wall is removed. While the joint was fixed, this moment was resisted by the wall. After release, it must obviously be resisted by an equal and opposite moment in the beam. Hence, the release and subsequent rotation of the joint causes no change in M_{ao}, but adds a -1000 to M_{ab}. This value is written as shown in Fig. 12. Joint A is now said to be balanced, since $\Sigma M = 0$.

(2) Carry-over moment to joint B: The balancing of joint A actually amounted to applying a moment of -1000 to end A of beam AB. This -1000 deflected the center of the beam upward (Fig. 12) and also caused a moment to be "induced at" or "carried over" to M_{ba}, since joint B was fixed against rotation. This moment change at M_{ba} is equal to $+1/2 \times -1000 = -500$.

(3) Joint A is assumed fixed in its new position.

(4) Balance joint B: It will be noticed that this joint differs from any encountered so far in that its rotation affects and is affected by a beam on each side. The instant after joint B is released, the unbalanced moment is $M_{ba} + M_{bc} = -500$. In order that the moments on each side of the joint will be equal and opposite, the equilibrant of the unbalanced moment of
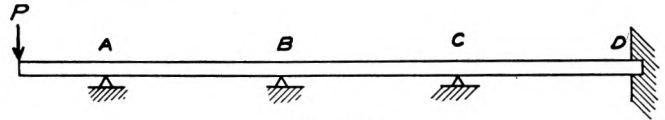
— Fig. 10 —
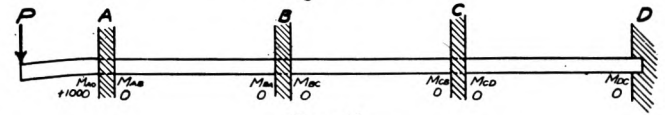
— Fig. 11 —
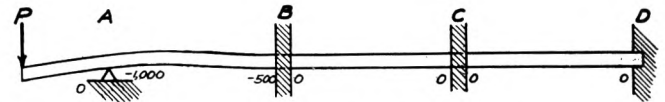
— Fig. 12 —
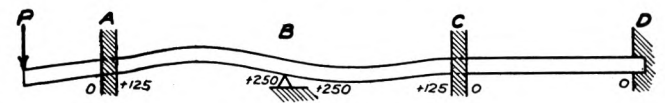
— Fig. 13 —
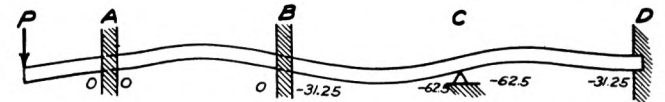
— Fig. 14 —
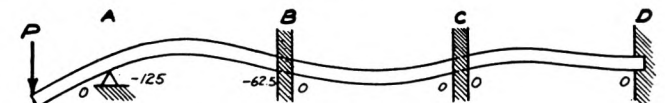
— Fig. 15 —

— Fig. 16 —

— Fig. 17 —
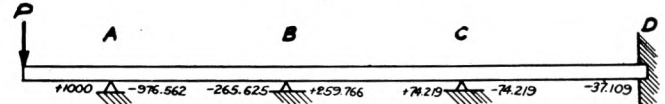
— Fig. 18 —

-500 must be distributed to each beam so that the equation $\Sigma M = 0$ is satisfied. That is, the joint rotates clockwise, relieving M_{ba} of some moment and adding an equal amount to M_{bc}.[13] If M_{ba} is relieved by $+250$ and the same amount is added to M_{bc}:

$$M_{ba} = -500 + 250 = -250$$

---

[13] This is true only when I/L is the same for the member on each side of the joint. Also, one-half is given to each member only when there are but two members coming into the joint. For further development and discussion of balancing moments at joints, see under heading, "Effect of Rigidity on Distribution of Moments at a Joint."

$$M_{bc} = 0 + 250 = +250$$
$$\Sigma M_b = 0$$

Hence, the equation of equilibrium for the joint is satisfied by distributing a moment equal to one-half the unbalanced moment and of opposite sign to the member on each side of the joint. In this case, the changes in M_{ba} and M_{bc} are recorded as shown in Fig. 13; but at this time the new values will not be added to the moments ($M_{ba} = -500$ and $M_{bc} = 0$) already existing.

(5) Carry-over moment to joints A and C: The balancing of joint B amounted to applying new moments of $+250$ at M_{ba} and M_{bc}. Hence, a moment of $+\frac{1}{2} \times 250 = +125$ must be carried over from M_{ba} to M_{ab} and from M_{bc} to M_{cb}.

(6) Joint B is assumed fixed in its new position.

(7) Balance joint C: The unbalanced moment at this joint is $+125$. The procedure of releasing and balancing is the same as for joint B in step (4). The results are shown in Fig. 14.

(8) Carry-over moment to joints B and D: Procedure same as in step (5). Carry-over moments are -31.25 to M_{bc} and -31.25 to M_{dc}, as shown in Fig. 14.

(9) Joint C is assumed fixed in its new position.

(10) Balance joint D: This joint is actually fixed and cannot be released. Hence there is no unbalanced moment to be distributed and the carry-over moment to $M_{cd} = 0$.

We have now completed one cycle and can return to joint A and follow through the same procedure from A to D again. It will be noticed that, when the joints are now released, the unbalanced moments are the algebraic sum of the carry-over moments from the preceding cycle. That is, the unbalanced moment at A (Fig. 15) is the $+125$ carried over when joint B was balanced in Fig. 13. The unbalanced moment at B (Fig. 16) is the algebraic sum of -31.25 carried over when joint C was balanced in Fig. 14, and -62.50 carried over when joint A was balanced in Fig. 15. The total is -93.75. To balance the joint, a $+46.875$ must be distributed to each member.

In this example, the computations are carried through three cycles. In each step, only the moments resulting from that particular change are recorded. The final moment at any joint must equal the fixed-end moment with which we started, increased or decreased by the various moment increments resulting from the separate steps. The totals are as shown in Fig. 18. Further cycles would bring the moments on either side of a support into closer agreement. Inspection of the results indicates that it is not necessary to follow the exact procedure used. It is more practical to use the method and form of computing shown in Fig. 19. The fixed-end moments are first

written for all joints—step (a) in the figure. As previously explained, all fixed-end moments are zero in this case, except on the cantilever. Joint A is then balanced and a -1000 distributed to the right of the support and zero to the left—step (b). The carry-over moment to B is not recorded at this time. Joints B, C, and D are balanced next, thus completing step (b). Since the fixed-end moments at these joints are equal to zero there is nothing to distribute. A line is drawn under the value obtained from step (b). These lines do not necessarily signify an addition; they are merely convenient marks to indicate that the joint is in balance. At this stage the sum of the moments on either side of a joint should equal zero since the carry-over moments have not yet been computed. The carry-over moments resulting from step (b) are now computed for all joints, and the values are recorded under the lines—step (c). After this step, certain joints are out of balance, so step (b) is repeated and the second set of lines drawn. Thus steps (c) and (b) are repeated as many times as desired. In the illustration, the computations are carried through six cycles before the columns are added to obtain the total moments—step (t). A simplified attack involving fewer computations for this type of problem will be illustrated later.

The reader might at this point question the correctness of computing the total moments after balancing joints rather than after recording carry-over moments. While the latter way would be more nearly correct, the quantities involved are small in any event. The method used in the illustration has the advantage that an exact check can be obtained; i.e., the sum of the moments on each side of a joint equals zero. This check obviously does not apply to joint D.

The foregoing examples illustrate the application of the Hardy Cross method to very elementary structures. To summarize: A structure is taken without any external loading; all joints, including those normally free to rotate, are assumed as fixed against rotation; the external loads are then applied, and the resulting fixed-end moments are computed by well-known methods. In succession, each joint is released, allowed to rotate as far as it can, and then fixed again in its new position. This rotation allows the moments to balance at the joint which is released, but further adds to the unbalance of the joint at the far end of each rotated member. It is analogous to making several adjustments to a machine where a change in any one adjustment throws the others out again. The closer the structure gets to its final deformed state, the smaller the rotations and, hence, the smaller the carry-over moments causing unbalance at

FIG. 19

adjacent joints. The general method is one of assuming the structure to deflect to its natural loaded position through a series of steps. Each step is paralleled by a computation of the moments needed to balance the joints released and the resulting carry-over moments imposed on adjacent joints. The moment changes at each step are evaluated before proceeding to the next step. By the principle of superposition, the results of the various steps are added algebraically to determine the final moments. The accuracy of these final moments depends on the number of steps or cycles used in the computations.

The general physical action upon which the Hardy Cross method depends is aptly illustrated by a building frame as shown in Fig. 20. All joints of the frame are free to rotate if desired, but it is assumed that they are fixed as far as translation in any direction is concerned. Suppose a moment acts on joint 1 when all other joints are assumed as fixed against rotation as well as lateral movement. As a result, certain fixed-end moments are set up at 2, 3, 4, and 5 due to the carry over from joint 1. These carry-over moments will be unbalanced moments at the instant these joints are released. If all joints except 2 are now fixed, and joint 2 is released to that $\Sigma M_2 = 0$, this release will in turn cause a further spread of moment, not only to new joints, but also back to joint 1. The same effect will result from releasing joints 3, 4, and 5 in turn. It will be noticed that the releasing of joints 2, 3, 4, and 5 affects some joints in common (Fig. 21).

If joint 10 (Fig. 22) is now released, joints 2 and 3 are affected, as well as some new joints. Hence, it might be said that as the wave spreads out from the origin, it not only affects new joints in its path, but it also affects the joints in its wake. As it recedes from the point of origin, its intensity decreases until it "dies out." A new wave of less intensity now moves out from the origin, leaving still smaller moments in its wake. Successive waves grow smaller until their effect is practically negligible.

EFFECT OF RIGIDITY OF MEMBERS ON JOINT ROTATION

It has been stated under "Definitions" that the stiffness or rigidity of a straight beam of constant section is proportional to the moment of inertia divided by the span length. That is, the value of $K = I/L$ is an index of rigidity.

It will be recalled that the continuous beam used for illustration, was of constant section and that the spans were of equal length. Hence the value of K was the same for all members. Also, in balancing joint B (see item 4 under heading "Illustration, Continuous Beam") it was stated that the equation of equilibrium for the joint was satisfied by distributing a moment equal to minus one-half the unbalanced moment to the member on each side of the joint. However, it should be observed that this was true only because the beams on either side of joint B were equally stiff. Since $K_{ba} = K_{bc}$, the distribution was actually in the ratio of $\frac{K_{ba}}{K_{ba} + K_{bc}} M_u = \frac{K_{ba}}{2K_{ba}} M_u = \frac{1}{2}M_u$, where M_u is the unbalanced moment. If $K_{ba} = 2$ and $K_{bc} = 3$, the amount distributed to M_{ba} and M_{bc} would be $\frac{2}{2+3} = \frac{2}{5}$ and $\frac{3}{2+3} = \frac{3}{5}$, respectively. That is, the distribution of the unbalanced moment to the members was in proportion to their relative rigidities.

The physical effect of relative rigidity can best be seen by following through the various steps illustrated in Figs. 23 to 28.

Fig. 23 shows a two-span continuous beam with a load P on one span, a knife-edged support at B, and fixed at ends A and C. The knife-edge is assumed to have no restraining effect on joint B so far as rotation is concerned. Suppose joint B is assumed fixed while the load P is being applied and is then released— the usual procedure in the Hardy Cross method. If, as in Fig.

24, beam BC is infinitely stiff compared to beam AB, obviously there will be no rotation at joint B when the fixity is removed and the moment M_{ba} will be the same as the original fixed-end
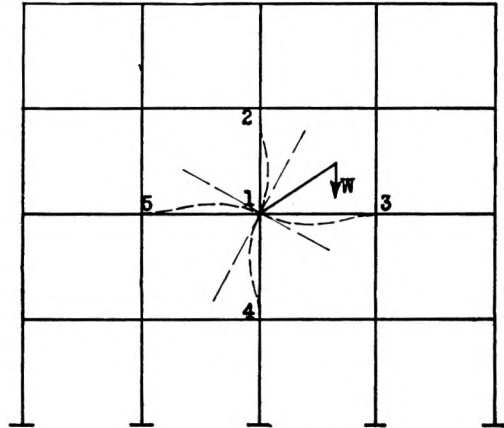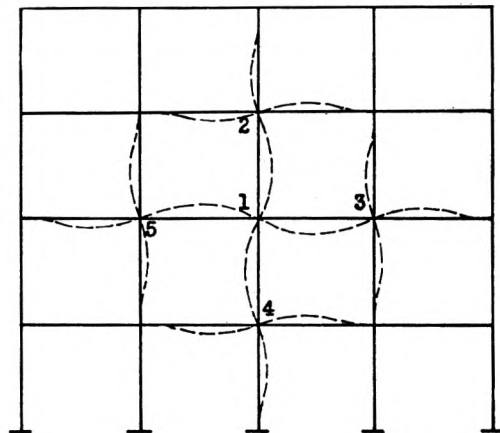

FIG. 20
(First stage, joint 1 rotates.)

FIG. 21
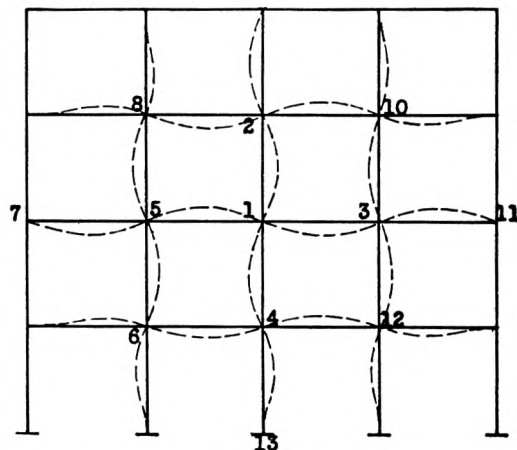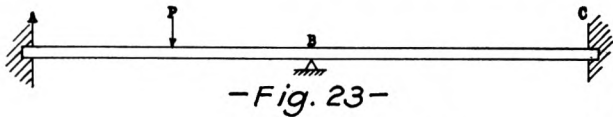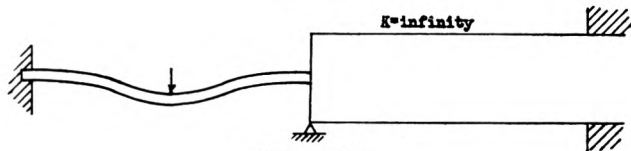(Second stage, joint 1 fixed, joints 2 to 5 rotate.)

FIG. 22
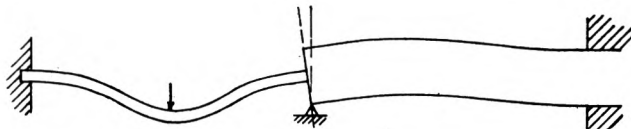(Third stage, joints 1 to 5 fixed, joints 6 to 12 rotate.)

moment. If, as in Fig. 25, beam BC is less rigid but still relatively stiff as compared to AB, some rotation of B will take place upon release and the resulting moment M_{ba} will be less

-Fig. 23-

K=infinity

-Fig. 24-

-Fig. 25-

-Fig. 26-

-Fig. 27-

K=0

-Fig. 28-

E is a constant, if both spans are of the same material, and if we let $I_1/L_1 = K_1$ and $I_2/L_2 = K_2$:

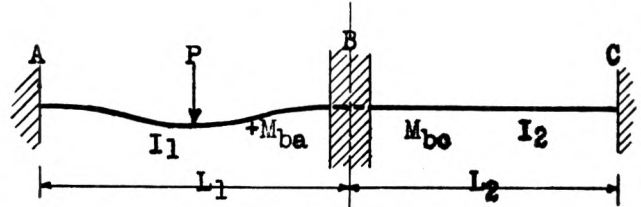$$M'_{ba}/K_1 = M'_{bc}/K_2 \dots \dots \dots \quad [1]$$
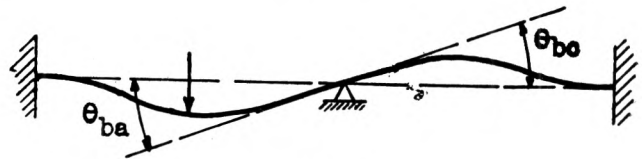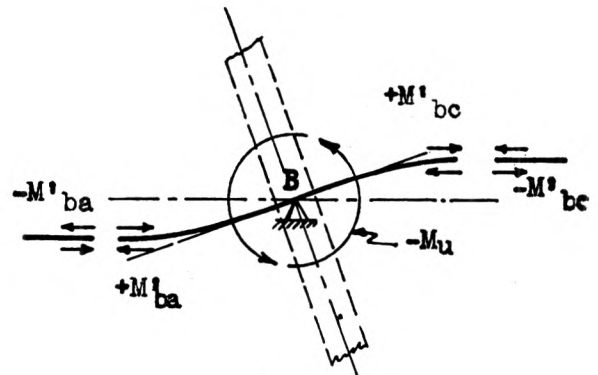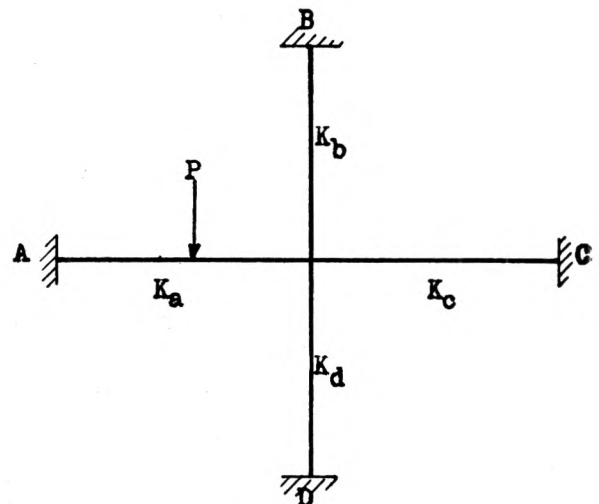

Fig. 29

Fig. 30

Fig. 31

Fig. 32

than the original fixed-end moment. Figs. 26 and 27 show the effect of further reduction in the stiffness of beam BC. If, as in Fig. 28, beam BC ceases to exist or is of infinitely small cross-section, joint B can rotate until $M_{ba} = 0$. Hence, it is seen that the rotation of a joint is largely dependent on the relative rigidities of the members common to the joint.

EFFECT OF RIGIDITY ON DISTRIBUTION OF MOMENTS AT A JOINT

Let us now take the same beam as was used in Fig. 23, designate the moment of inertia and span length as shown in Fig. 29, and apply the first step of the Hardy Cross method—fix joint B and apply the load P. As a result $M_{bc} = 0$; but M_{ba} is a fixed-end moment depending on the load and span length. This fixed-end moment is held in equilibrium by the clockwise couple exerted by the hypothetical wall at B. Let us call this $+M_u$. If the wall is suddenly replaced by the knife-edged support, the moment $+M_u$ is momentarily unbalanced until it is removed. Removing a moment is the same as applying an equal and opposite moment. Hence, it can be considered that the unbalanced moment $+M_u$ is removed by the application of an equal and opposite moment $-M_u$. This moment rotates the tangent to the elastic curve of the beam through the angle $\theta_{ba} = \theta_{bc}$ (Fig. 30) and causes corresponding changes in the beam moments. Let us call the moment changes due to the rotation M'_{ba} and M'_{bc}. From the moment area or elastic-weights method of determining moments:[14] $\theta_{ba} = M'_{ba}L_1/4EI_1 = M'_{bc}L_2/4EI_2 = \theta_{bc}$, or, since

So as to have equilibrium at the joint after rotation (Fig. 31):

$$M'_{ba} + M'_{bc} = -M_u \dots \dots \dots \dots [2]$$

Therefore, from Equations [1] and [2]:

$$M'_{ba} = -M_u \times K_1/(K_1 + K_2) = -M_u \times K_1/\Sigma K$$
$$M'_{bc} = -M_u \times K_2/(K_1 + K_2) = -M_u \times K_2/\Sigma K$$
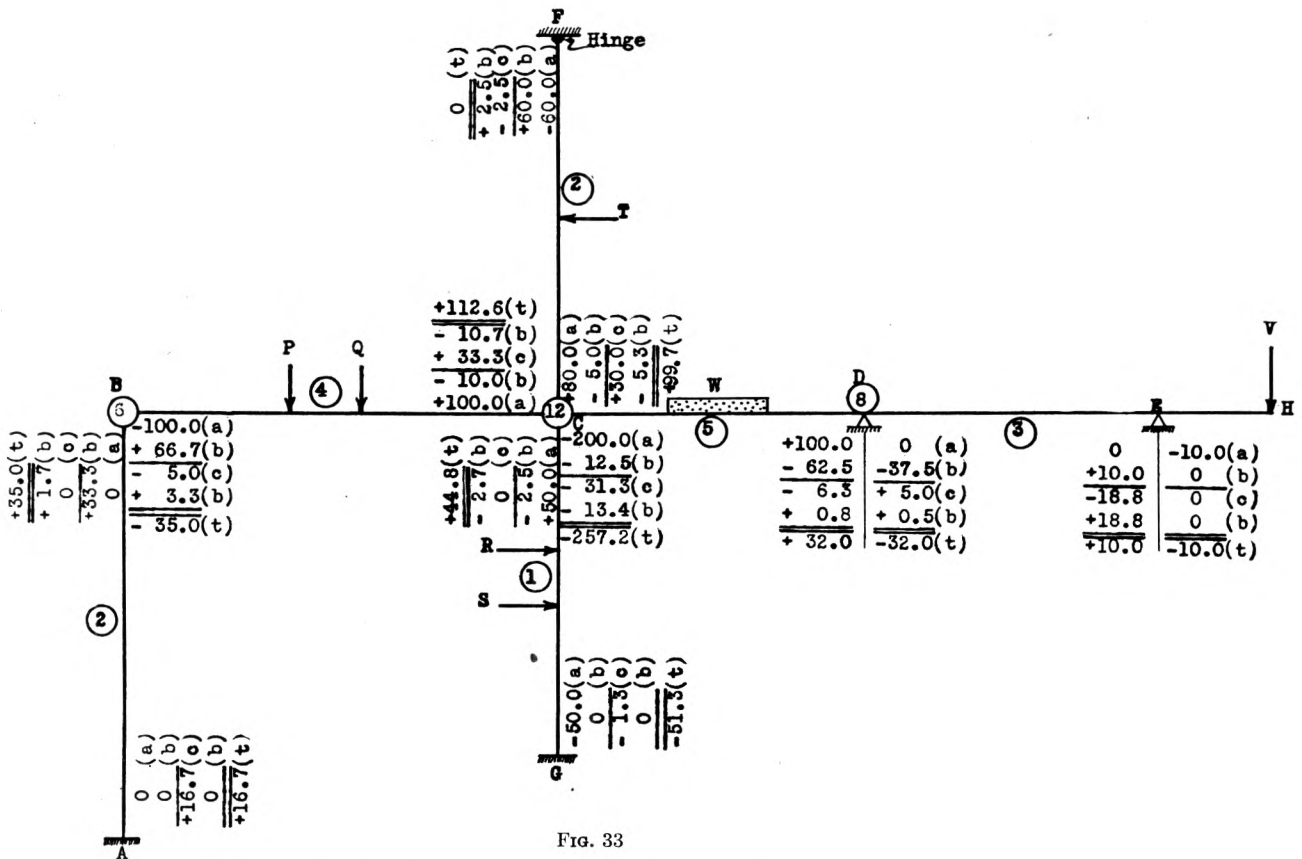
[14] See Appendix B.

FIG. 33

Attention is again called to the fact that M'_{ba} and M'_{bc} are only the moments due to the rotation of joint B after the fixity is removed. The actual total moments in the two members are found by adding the moments M'_{ba} and M'_{bc} algebraically to the original fixed-end moments M_{ba} and M_{bc}, respectively.

In so many words, the foregoing equations tell us that when a joint is released, a moment equal and opposite to the unbalanced moment is distributed to the members common to the particular joint in proportion to their rigidities. If the unbalanced moment is positive, the moments distributed will be negative, and vice versa.

If there are more than two members common to the joint released (Fig. 32), the rule still holds. That is:

$$M'_{ba} =$$
$$-M_u \times K_a/(K_a + K_b + K_c + K_d) = -M_u \times K_a/\Sigma K$$

APPLICATION OF HARDY CROSS METHOD TO AN ELEVATOR SPAR

Fig. 34 shows the vertical-load curve for an elevator spar.[15] The spar will first be considered as a symmetrical continuous beam with five supports. A simplified attack will then be shown.

If a $1\frac{1}{4}$-in. by 0.035-in. Dural tube is used, the moment of inertia is 0.02467 throughout, and the value of K will be the same for all spans except the overhangs, where it will be zero. Hence, a relative value of $K = 1$ is used.

All joints are assumed fixed against rotation, and the fixed-end moments are calculated:[16]

[15] The author is indebted to Mr. Ben W. James, of Stanford University, for the numerical data used in this illustration.

[16] The expressions for determining fixed-end moments can be derived by moment areas or slope deflection as shown in Appendix B. Values for various types of loadings are given in Fig. 41.

$$M_{Fao} = -M_{Fef} = 1.32 \times 5 \times 10/3 = 22$$
$$-M_{Fab} = +M_{Fed} = \frac{1.32(45)^2}{12} + \frac{0.96(45)^2}{2 \times 15} = 288$$

Similarly:

$$M_{Fba} = -M_{Fde} = 320$$
$$-M_{Fbc} = +M_{Fdc} = 450$$
$$M_{Fcb} = -M_{Fcd} = 482$$

These quantities are recorded as shown on Fig. 35, step (f). In accordance with the conventions given under "Definitions," clockwise-resisting moment couples are positive, counter-clockwise couples are negative. The subsequent procedure is as follows:

Joint A: The instant after this joint is released, the summation of moments around the joint is $+22 - 288 = -266$. To make the joint balance, a moment of $+266$ must be distributed to the two members in proportion to their rigidities (see heading, "Effect of Rigidity on Distribution of Moments at a Joint").

Hence, distribute to AO, $\dfrac{0}{0+1} \times +266 = 0$; to AB $\dfrac{1}{0+1} \times +266 = +266$. These quantities are recorded as shown (step b) and a line is drawn under each to indicate that the joint is now in balance, i.e., $\Sigma M_A = 0$. As stated before, this line does not indicate an addition.

Joint B: The instant after this joint is released, the unbalanced moment is $+320 - 450 = -130$. The subsequent rotation will be the same as if a $+130$ acted on the joint. Hence, distribute $1/(1 + 1) \times +130 = +65$ to BA and the same to BC.

Joint C: There is no unbalanced moment at this joint because of symmetry.

Joints D and E: Balance and record in the same way as for joints B and A, respectively. Note that the signs are reversed on this side.

All joints are now in balance. Since the balancing actually consisted in allowing one joint at a time to rotate while all others were fixed, the resulting carry-over moments must now be computed as indicated in step (c). As has been previously shown, the carry-over moment at one end of the member is the distributed moment just determined at the opposite end multiplied by $+\frac{1}{2}$.

The moments just carried over throw the joints out of balance again. These new moments must be distributed in the same way

cal action of the beam when A is released while B is assumed fixed. Now balance joint B. The unbalanced moment is $+320 + 133 — 450 = +3$. Therefore, distribute —3 as shown, 3/7 to BA and 4/7 to BC, and at once record the corresponding carry-over moment at C.

There will be no carry-over to A, since this joint was free to rotate. It will be noticed that the solution is completed, since C need not be released.

This procedure can be followed with more complicated structures even when symmetry and freely supported joints are not involved.

In most structures, there will be certain joints with a larger
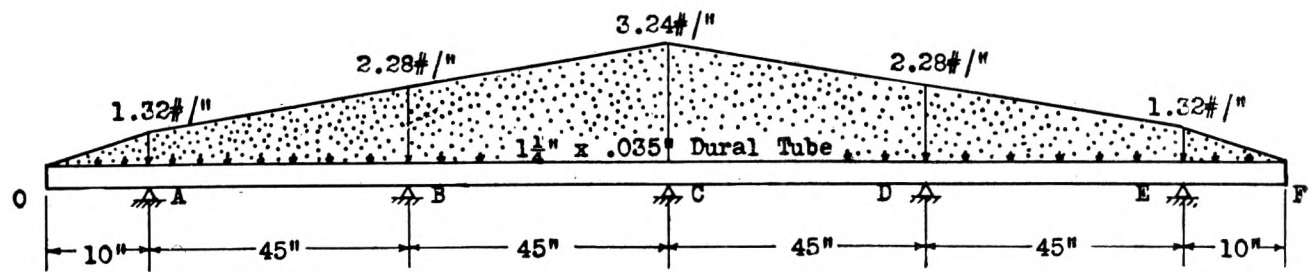

FIG. 34

FIG. 35

as the fixed-end moments. In Fig. 35, the computations are carried through four cycles, after which a double line is drawn and the algebraic totals are written.

Other quantities such as moments, shears, and reactions can be found now by the usual laws of statics.

SOLUTION SIMPLIFIED

The foregoing computations for the elevator spar can be greatly simplified. Because of symmetry, only one-half of the beam need be considered, as shown in Fig. 36. The center joint C can be assumed fixed, since we know it does not rotate on account of the symmetry of structure and loading. Joint A is freely supported and, once released, need not be fixed again. This necessitates the use of a value for K_{ab} equal to 3/4, the true relative value. Hence, to simplify computations, K_{bc} is made equal to 4 and K_{ab} equal to 3.[17]

Joint A is now balanced as before, and $+133$ is immediately carried over to joint B. This procedure is more nearly in accord with the physi-

[17] It can be shown by slope deflection or moment areas that the moment needed to produce a given rotation at one end of a beam when the other end is free is three-fourths as great as if the other end is fixed.

unbalanced moment than others. Where the unbalanced moment is quite small, it is a waste of effort to balance the joint carefully, only to have this balance entirely disrupted later by a large carry-over moment from adjacent joints. Hence, more rapid convergence will be obtained if the highly unbalanced joints are first balanced and the resulting carry-over moments are recorded before proceeding to the better balanced joints.[18]

[18] This method was proposed by L. E. Grinter in discussion of the Hardy Cross paper. See Trans. Am.Soc.C.E., vol. 96, 1933, p. 11. It is suggested that this method should not be used until complete familiarity with the usual procedure is obtained.

FIG. 36

APPLICATION TO ELEVATOR SPAR WITH DEFLECTED SUPPORTS

The elevator spar of Fig. 34 is attached to the stabilizer. Let us suppose that it has been determined from computations or from tests that the supports at A, B, D, and E are deflected by known amounts with respect to C. The moments in the spar can be found by moment distribution as before, once the revised fixed-end moments are determined.

It will be recalled that the first step in the Hardy Cross method is to assume all joints fixed and then apply the external loads. If this fixity is also assumed to take place before the joints are deflected, this deflection will set up fixed-end moments in any one span, as shown in Fig. 37.
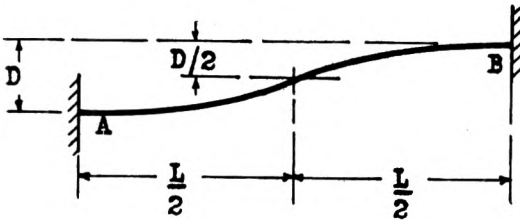

FIG. 37

FIG. 38

If the member AB is of constant cross-section, the point of contraflexure will be at the mid-point and each half can be treated as a cantilever beam with an end deflection of D/2.

Then:

$$\frac{D}{2} = \frac{PL^3}{3EI \times 8} = \frac{ML^2}{12EI} \quad (L = \text{total length of member})$$

or:

$$M = \frac{6EI}{L^2} D$$

Let us suppose it has been found that A deflects downward $^1/_8$ in. with respect to B, and B deflects the same with respect to C. The deflections of E and D are the same as A and B, respectively. For a $1^1/_4$-in. by 0.035-in. Dural tube, $I = 0.02467$ and $E = 10,400,000$. Then the foregoing formula gives the following fixed-end moments due to deflection of supports alone:

$$M_{ao} = 0$$

$$M_{ab} = \frac{6 \times 10,400,000 \times 0.02467 \times 0.125}{45 \times 45} = 95$$

and

$$M_{ba} = M_{bc} = M_{cb} = 95$$

The sign of this moment must be determined by inspection. If A deflects downward with respect to B, as in Fig. 37, obviously a positive clockwise moment at each end results. When these values are added algebraically to the fixed-end moments caused by the external loading (Fig. 35), the results are as shown in

Fig. 38, step (f). The succeeding steps in the solution are the same as for the example of Fig. 36.

PRACTICAL APPLICATION OF THE METHOD TO A CONTINUOUS FRAME

A more general use of the method will now be illustrated by determining the moments in the continuous frame shown in Fig. 33. This is the same frame that Professor Cross used in his original paper. As he stated at that time, the illustration is entirely academic and is selected only because "it involves all of the conditions that can occur in a frame which is made up of straight members and in which the joints are *not displaced*." In Fig. 33, the sign convention has been revised to agree with that defined at the beginning of this paper. Other minor changes from the original paper, suggested in the subsequent discussions, have been adopted where it was felt that the chance of error would be lessened. No attempt has been made to introduce short cuts in this solution. Joints E and F could obviously be treated like joint A in Fig. 36.

The relative $K = I/L$ value for each member is shown by the number in the circle near the middle of the span. The sum of the K's for all members common to a joint is shown in the circle at the joint.

It is assumed that all joints are fixed, and then the loads P, Q, R, S, T, W, and V are applied as shown. The resulting fixed-end moments are then determined and are recorded, as shown in the figure, parallel to the member concerned and followed by the letter (a) for reference.

The unbalanced moment at all joints is first distributed, step (b). It will be noticed that the algebraic sum of the fixed-end moments at joint C is +30.0. This is the unbalanced moment the instant after the joint is released. The subsequent rotation will be the same as if a moment of —30.0 acts on the joint. The distribution will be 4/12 to CB, 2/12 to CF, 5/12 to CD, and 1/12 to CG, as shown. The subsequent procedure for this and other joints should be apparent in view of the previous illustrations.

Only two cycles are used in this example. Further cycles will improve some of these results, especially at joint D. The probable effect of further computation can always be ascertained by inspection of the last set of carry-over moments.

PRACTICAL APPLICATION OF HARDY CROSS METHOD TO DESIGN OF AIRPLANE FUSELAGE TRUSS

Fig. 39 shows the forward section of an airplane fuselage truss. Member 2U-4U has a concentrated load of 62 lb located 13 in. from one end, due to instruments. Members 3L-4L and 4L-5L are subjected to a uniform floor load in addition to concentrated loads between panel points due to passengers. Member 5L-6L has a concentrated baggage load of 310 lb located as shown.

The sizes, lengths, and axial loads determined in the usual way,

TABLE 1

Member	Size, in.	Length, in.	Axial Load, lb
2U-4U	$1^1/_4 \times 0.035$	58	+3260
4U-5U	1×0.035	48	—2600
5U-6U	1×0.035	37	—3860
6U-7U	$1^1/_8 \times 0.035$	50	—3865
2L-3L	1×0.035	40	—2830
3L-4L	$1^1/_4 \times 0.035$	39	—5340
4L-5L	$1^1/_4 \times 0.035$	26	—6733
5L-6L	1×0.035	36	—3700
6L-7L	1×0.035	51	—2650
2U-3L	1×0.035	54	—2065
3L-4U	$1^1/_8 \times 0.035$	57	—2520
4U-4L	$1^1/_8 \times 0.035$	58	—1475
4L-5U	$1^1/_4 \times 0.049$	60	—4700
5U-5L	$1^1/_4 \times 0.049$	54	—5216
5U-6L	1×0.035	65	—1200
6U-6L	$^7/_8 \times 0.035$	60	— 870
6U-7L	$^7/_8 \times 0.035$	71	+ 550

assuming pin joints for each member, are shown in Table 1 (the axial loads are the critical values based on a combination of high angle of attack and three-point landing conditions).

Under present practise, a truss with this type of loading would be analyzed by assuming the members which have no side loads to act as struts with ends restrained in such a manner that the restraint coefficient $c = 2$ could be used. For the members subjected to side loads between joints, the Department of Commerce requires that the bending moments be properly taken into account, but does not specify how this should be done.

This paper proposes a method of attack for analyzing the effect of these intermediate loads which will give definite results even though it is not claimed that they will be perfectly accurate. It is believed to be an improvement over the present uncertainty and to constitute the first step toward obtaining a practical rational solution of the problem.

The most important errors in the method are due to the following factors:

(a) Bending moments due to joint deflection, resulting from changes in length of members, are neglected. (See discussion of Hardy Cross paper by Thompson and Cutler, Trans. Am. Soc. C.E., vol. 96, p. 108, for a method of obtaining these moments.)

(b) An axial compression load acting on the ends of a continuous beam causes a deflection in each span. These deflections in themselves result in additional moments at supports. These moments are neglected.[19]

(c) To get the allowable stress in compression members from Fig. 11 of Aeronautics Bulletin, No. 7-A, edition of Jan. 1, 1932, arbitrary assumptions are made regarding the location of points of inflection in selecting the value of L.

(d) To reduce labor, only a part of the truss is considered, and the end joints of this portion are assumed as fixed against rotation. This is a minor error.

The proposed procedure is made up of the following steps:

1 Compute the primary bending moments in all members due to loads between panel points by the Hardy Cross method. If desired, the secondary moments listed under (a) could be included in this computation, though that is not done in the illustrative example.

2 Determine maximum bending moment and unit stresses due to bending and axial load.

3 Determine effective slenderness ratio and compute margin of safety from Figs. 10 to 12 of Bulletin 7-A.

The determination of the maximum bending moment is likely to lead to some difficulties. The present practise of using a restraint coefficient of 2.0 in designing a fuselage member implies that the magnitudes and directions of the end moments are such that there is a point of inflection

Fig. 39

---

[19] Since this paper was written, Mr. Ben W. James, of Stanford University, has worked out a method of moment distribution modified to include the effect of axial loads upon the bending moments.
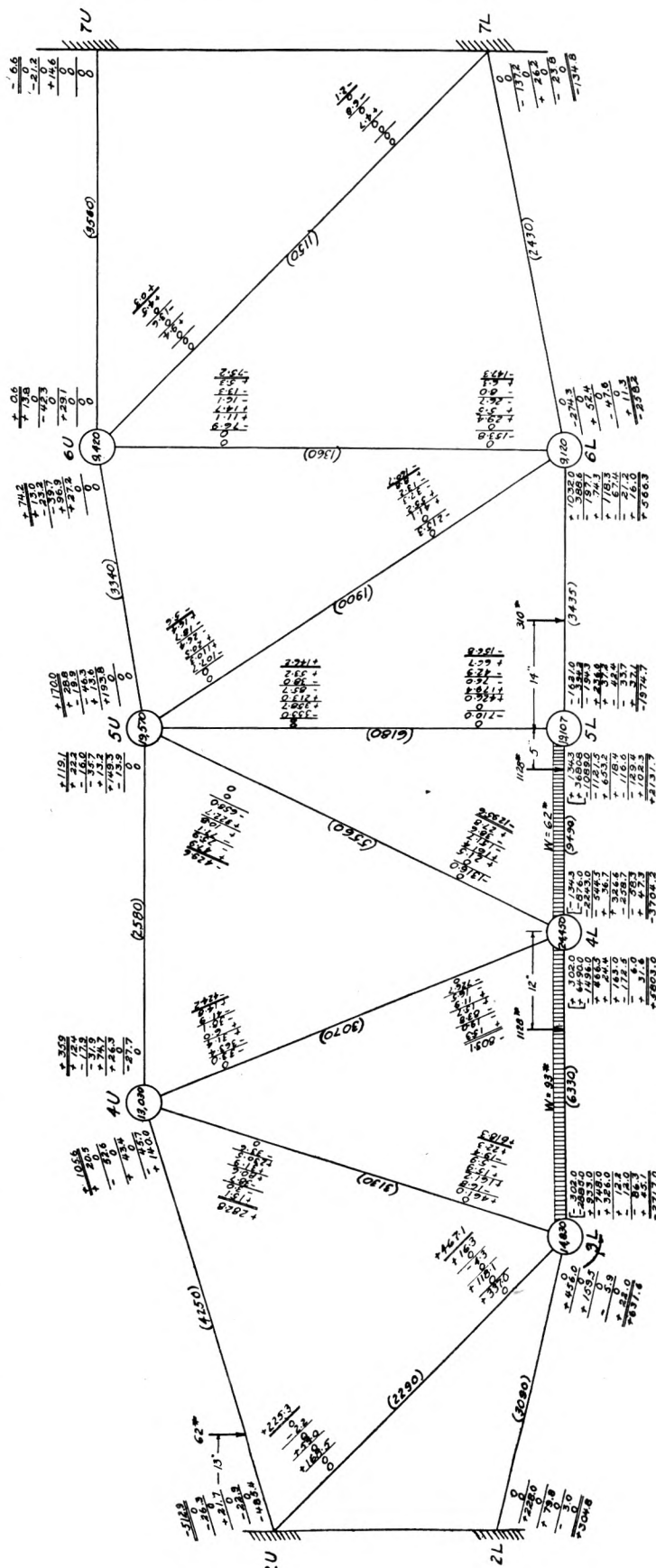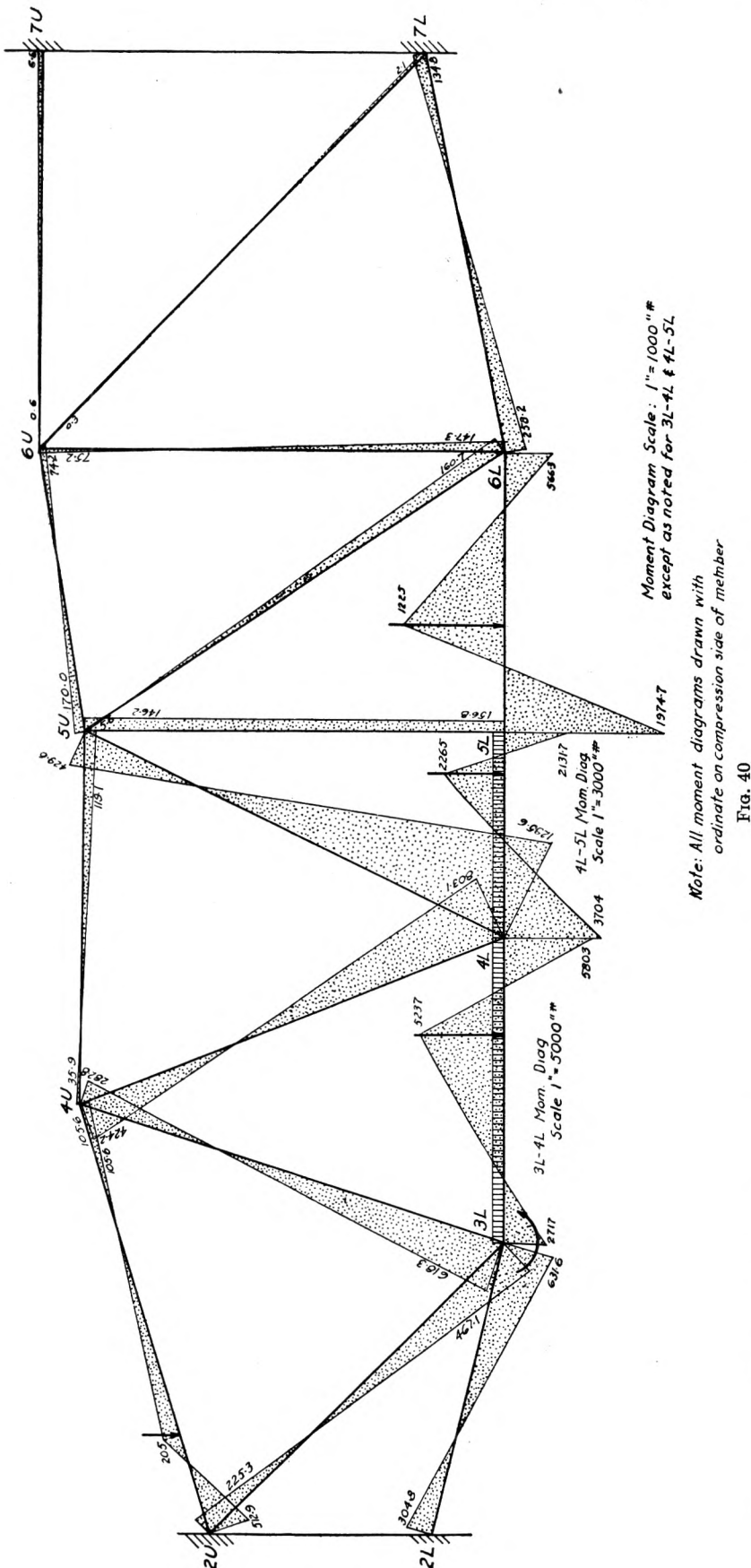
Moment Diagram Scale: 1" = 1000"#
except as noted for 3L-4L & 4L-5L

FIG. 40

Note: All moment diagrams drawn with
ordinate on compression side of member

distant not more than 0.7 L from the end of the member. As a result, if a member is designed on this basis with a small margin of safety for axial load only, the value of L/j will very likely be greater than π. If the moments at the ends of such a member were independent of the deflections of the member (as in a single span with cantilever ends as shown in Fig. 11:12 of "Airplane Structures" by Niles and Newell), the axial load would be greater than the critical and the member unable to carry it. A member of a fuselage truss, however, is more like one span of a continuous beam, such as is shown in Fig. 11:13 of "Airplane Structures." The end moments are very much influenced by the deflections in the member, and the critical load is not that which gives $L/j = \pi$, but a larger value that would be obtained by computations similar to those in article 11:7 of "Airplane Structures."

The rational method of computing the end moments would have to include computations of the effect of the secondary deflections between panel points to be made by some suitable modifications of the extended three-moment equation of chapter 11 of "Airplane Structures," but it has not yet been found possible to accomplish this. In the meantime, it is necessary to make certain assumptions.

For members in which the axial load is tension it seems conservative to neglect the effect of combined bending and axial load in producing secondary moment. The unit stress due to axial tension may be added to that due to the bending computed by applying the Hardy Cross method and the sum compared to the allowable unit tension.

If there is no side load on the member and the Hardy Cross computations indicate that there is no point of inflection in the member, it would seem unwise to use a member designed on the basis of $c = 2.0$, but the design should be based on $c = 1$ so that L/j would be less than π, and the formulas of chapter 11 of "Airplane Structures" could be used with some confidence to determine the maximum moment in the span.

If there is no side load and the Hardy Cross computations indicate a point of inflection, the same practise could be followed, but it seems unnecessarily conservative. Owing to the continuity of the structure, the end moments will increase faster than the external loads and the point of inflection will probably have only a small movement. If this were not the case, the failing loads for fuselages subjected to static test would have indicated the necessity of designing for $c = 1$ or less, instead of $c = 2$. The designer has two reasonable assumptions he might make. He might assume the point of inflection to be 0.7 L from the end with the larger moment or that it was, say, one-third of the

distance from the point of inflection indicated by the Hardy Cross computations and the end with the smaller moment. In either case, the distance from the assumed point of inflection to the end with the larger moment would probably give a value of L_i/j less than π. This segment of the beam could then be analyzed in the usual manner with the formulas of "Airplane Structures" and the margins of safety computed by the system recommended in Bulletin 7-A.

When there is a side load on the span, there are two similar choices. The first is to use a member of such size that L/j is less than π and compute the maximum moment from the formulas of "Airplane Structures." The other is to find the points of inflection indicated by the Hardy Cross computations of end moment and assume points of inflection a little further apart.

In all of these methods, the end moments are assumed to be those computed by the Hardy Cross method, neglecting the secondaries due to the combination of bending and axial load in each member of the structure. This is a serious error, but one that cannot be eliminated until a method of applying the extended three-moment equation to a structure like a fuselage side truss has been developed. When that has been done, and efforts to this end are being made, a more rational solution of the problem can be developed. Until then, the method proposed here may be taken as a preliminary step in this direction and should give very reasonable results.

The chief alternative is to assume that all members subjected to side load have pin ends and to compute the bending moments in them by the formulas of "Airplane Structures." The objections to that procedure would be:

1 No computation would be made of the bending moments imposed on adjacent members without side load due to the continuity of the structure.

2 Members which probably have no point of inflection would not be detected, and would be designed for $c = 2$ instead of $c = 1$.

3 The effect of the end moments in reducing the bending moments near the center of the member would be neglected. Though this would be conservative, it would often be unnecessarily so.

These objections are quite as weighty as those against the proposed method.

In Fig. 39, the fixed-end moments are computed for the dead loads between panel points, and the moments at the ends of all members are determined from these by the usual Hardy Cross method. The results of those computations are used to draw the moment diagrams (Fig. 40) in which the ordinates are shown on the compression side of the member. The margins of safety are found for several representative members as shown below.

Member 2U-4U is subjected to an axial tension of $+3260$ and a concentrated side load. The maximum moment is -513 at the left end.

Hence:

$$f_b = \frac{My}{I} = \frac{513}{0.03948} = 12,980 \text{ lb per sq in.}$$

$$f_c = \frac{P}{A} = \frac{3260}{0.1336} = 24,400$$

$$f_t = 37,380 \text{ lb per sq in. tensile stress}$$

Assuming allowable stress for combined bending and tension $= 80,000$ lb per sq in.,

$$\text{Margin of safety} = \frac{80,000}{37,380} - 1 = +1.15$$

Member 6U-7U is a tension member. Obviously, the moment

at the right end of the member is maximum. Hence the procedure is the same as for 2U-4U, and the margin of safety is $+14.8$.

Member 5U-5L is subjected to a compressive load of -5216 lb and end moments as shown in Fig. 40. These end moments are opposite in rotation, so there are no points of contraflexure in the member.

From "Airplane Structures," we find $L/j = 3.96$ for a $1^{1}/_{4}$ by 0.49 member. This value is too large, since it is greater than π. Moreover, there are no points of inflection, so that we are not justified in using a length of less than 54 in. It would seem most logical to redesign the member so that $L/j < \pi$, as suggested.[20]

In practise it would be necessary to select a larger member and re-figure all end moments by the Hardy Cross method, since the larger strut would have a different stiffness factor.

Suppose we next try a $1^{5}/_{8}$ by 0.049 member, and assume for the purposes of this illustration that a redistribution of moments gives $M_1 = -165$ and $M_2 = -155$. (Transforming the signs to agree with those used in the formulas of "Airplane Structures.")

For these conditions, $L/j = 2.64$, and from case A of "Airplane Structures" it is found that the maximum moment of -645 in-lb occurs at a point approximately 27 in. from the bottom end of the strut.

At this point:

$$
\begin{aligned}
M_{\max} &= M_1 \sec x/j \\
&= -165 \times 3.91 &=& \quad 645 \\
f_b &= 645/0.0928 &=& \quad 6,950 \\
f_c &= 5216/0.2426 &=& \quad 21,500 \\
f_t &= 28,450 \\
f_b/f_t &= 6950/28,450 &=& \quad 0.244 \\
L/r &= 97
\end{aligned}
$$

Allowable stress from Fig. 11 of Bulletin 7-A $= 37,000$
Margin of safety $= 37,000/28,450 - 1 = +0.30$,
or $+30$ per cent

Member 4U-4L: This member is subjected to a compressive load and counter-clockwise moments at each end. The latter cause a point of inflection approximately 20 in. from 4U, as shown by Fig. 40, and it would seem reasonable to use $L_i = 38$ in. $+ 20$ in./3 $= 44.7$ in., or 45 in., say, measured from 4L. This is more conservative than using $0.7L$, since the latter would give $L_i = 40.6$ in.

For $L_i = 45$ in., $L_i/j = 2.4$, and since this is greater than $\pi/2$:

$$M_{\max} = \frac{M_2}{\sin L_i/j} = \frac{805}{0.6754} = 1190 \text{ in.-lb}$$

and

$$
\begin{aligned}
f_b/f_t &= 0.753 \\
L_i/r &= 116.5
\end{aligned}
$$

Allowable stress from Bulletin 7-A $= 51,000$

Margin of safety $= +2.2$ per cent

Member 4L-5L: This member is acted on by an axial compressive force, end moments of opposite rotation, a concentrated side load, and a uniformly distributed side load. For this member $L/j = 2.525$. Since this value is less than π, combined formulas from cases B and E of "Airplane Structures" may be used to compute the moments at several points and draw the moment diagram. Points on the moment curve are shown in Table 2, in which x is the distance from 4L.

Points of inflection occur at $x = 13.4$ and $x = 23.9$. The value of L_i is first assumed as 15 in. (estimated from the fact that $x =$

---

[20] A similar computation would also show the necessity of redesigning 3L-4L.
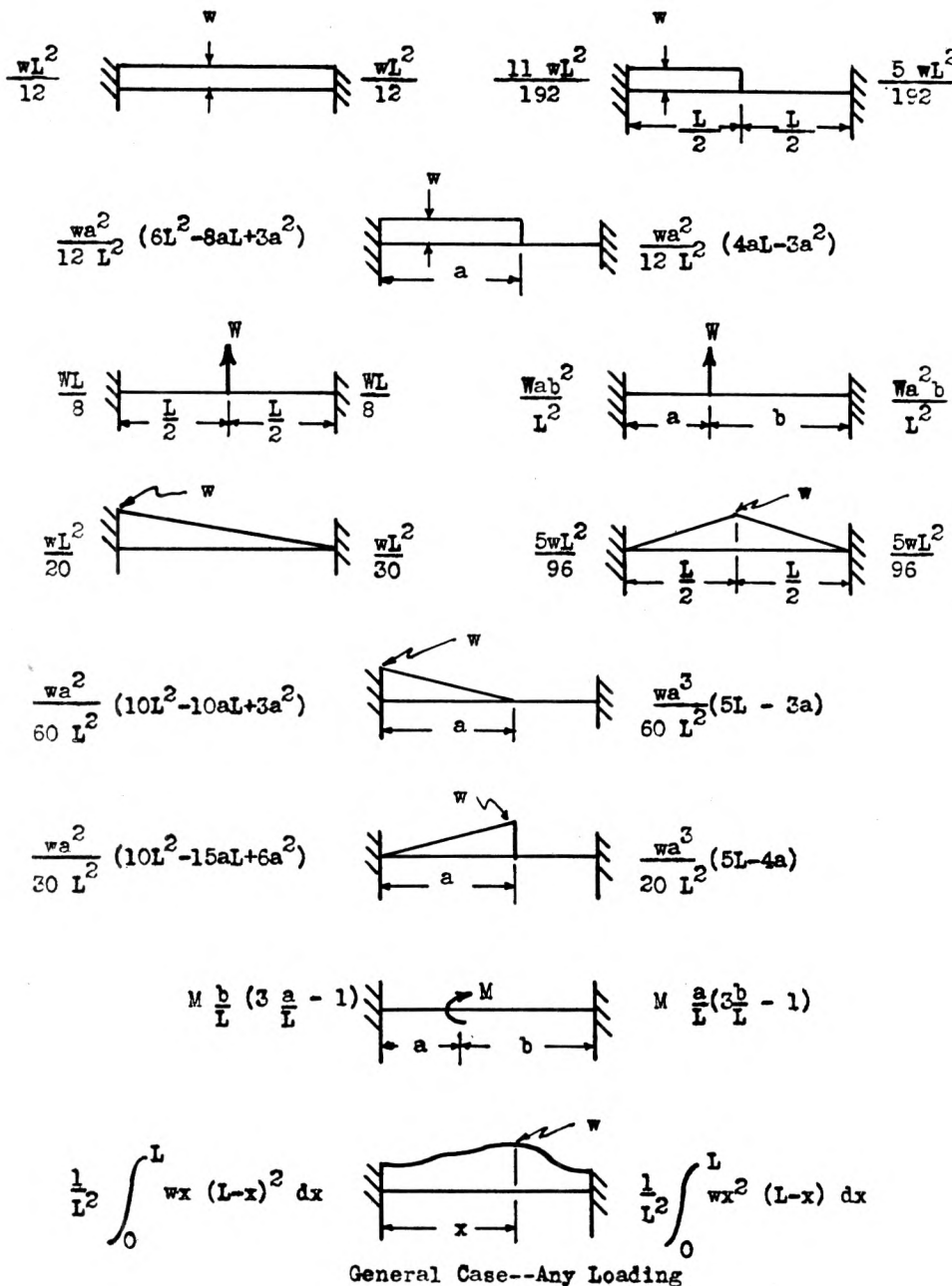
$$\frac{wL^2}{12} \qquad \frac{wL^2}{12}$$

$$\frac{11\,wL^2}{192} \qquad \frac{5\,wL^2}{192}$$

$$\frac{wa^2}{12\,L^2}(6L^2-8aL+3a^2) \qquad \frac{wa^2}{12\,L^2}(4aL-3a^2)$$

$$\frac{WL}{8} \qquad \frac{WL}{8}$$

$$\frac{Wab^2}{L^2} \qquad \frac{Wa^2b}{L^2}$$

$$\frac{wL^2}{20} \qquad \frac{wL^2}{30}$$

$$\frac{5wL^2}{96} \qquad \frac{5wL^2}{96}$$

$$\frac{wa^2}{60\,L^2}(10L^2-10aL+3a^2) \qquad \frac{wa^3}{60\,L^2}(5L-3a)$$

$$\frac{wa^2}{30\,L^2}(10L^2-15aL+6a^2) \qquad \frac{wa^3}{20\,L^2}(5L-4a)$$

$$M\frac{b}{L}\left(3\frac{a}{L}-1\right) \qquad M\frac{a}{L}\left(3\frac{b}{L}-1\right)$$

$$\frac{1}{L^2}\int_0^L wx(L-x)^2\,dx \qquad \frac{1}{L^2}\int_0^L wx^2(L-x)\,dx$$

General Case--Any Loading

Fig. 41

TABLE 2 POINTS ON THE MOMENT CURVE

x	Moment	
0	—3704	(end 4L)
5	—2795	
10	—1106	
15	+ 687	
21	+2490	(concentrated side load)
23	+ 717	
26	—2140	(end 5L)

13.4 to first point of inflection). The maximum moment for this case is —3704 at 4L. Using these values, it is found that the margin of safety is —39 per cent.

L_i is next assumed as 12 in., which is somewhat in excess of the distance between the two computed points of inflection. The maximum moment in this part of the span is +2490 under the concentrated side load. For these conditions the margin of safety is —25 per cent. Since these margins are negative, a heavier member should be used and new computations made throughout.

Member 5L-6L: This member is also a compression strut acted on by end moments of opposite rotations and a concentrated side load.

Since $L/j = 3.66$, which is greater than π, the method of attack used for member 4L-5L cannot be used. Owing to the continuity of the structure, it might be assumed that the points of inflection moved only one-third of the distance from their locations as indicated on Fig. 40, to the ends of the member. Then the central part of the member could be treated as a beam 25.6 in. long with a load of 310 lb located 8.25 in. from one end and with no end moments. For this beam L/j would be only 2.60, and the formulas of "Airplane Structures" could be used to determine the moment at the concentrated load. For this case we obtain $M_{max} = 432$ in-lb. Then

$$f_b = 432/0.02474 = 17,500$$
$$f_c = 3700/0.10611 = 34,800$$
$$f_t = 52,300$$
$$f_b/f_t = 0.34$$
$$r = 0.341$$
$$L_i/r = 75$$
$$F_t = 51,000$$

Margin of safety $= -0.024$,

or —2.4 per cent

This is evidently less conservative than the method used for member 4L-5L, since that method would have involved the use of a member with L/j less than π. It is difficult to say which method gives results more nearly approaching the facts of the case. Since there are seldom many members with side loads applied to them directly, it would seem desirable to use the more conservative method illustrated first.

It will be apparent that there is no necessity for using much accuracy in distributing moments by the Hardy Cross method, especially in the earlier trial designs. Even in the final design, the necessity of using more than three or four cycles is somewhat questionable.

The foregoing examples are intended as illustrations of the general method of attack. It is felt that this is a step in the right direction, and it is hoped that others will be sufficiently interested to develop it still further.

CONCLUSION

The Hardy Cross method of analyzing continuous frames by moment distribution has been received with widespread enthusiasm by structural engineers. It is hoped that some of this interest will spread to the aeronautical field. One aeronautical

engineer[21] has pointed out in discussion of the Hardy Cross paper that the method is of particular value for making rapid comparative studies of continuous closed frames, for assisting in digesting test results, and for solving other complex problems encountered in airplane design.

This method is not claimed to be the most practical for analyzing every structure encountered in design, but it is felt that its
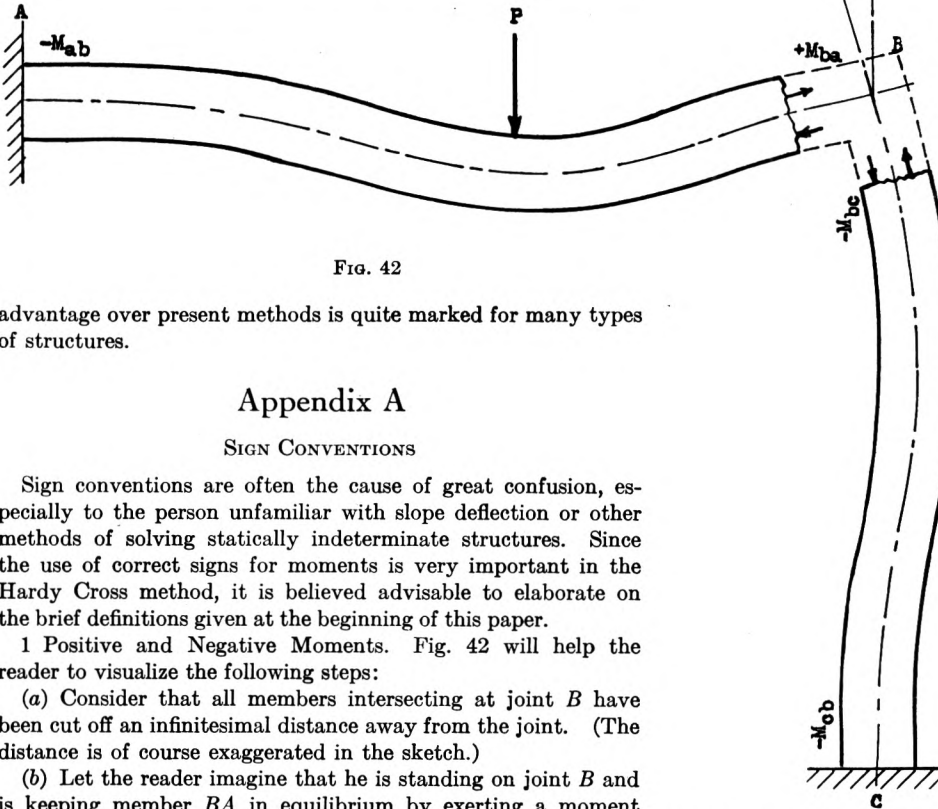

FIG. 42

advantage over present methods is quite marked for many types of structures.

Appendix A

SIGN CONVENTIONS

Sign conventions are often the cause of great confusion, especially to the person unfamiliar with slope deflection or other methods of solving statically indeterminate structures. Since the use of correct signs for moments is very important in the Hardy Cross method, it is believed advisable to elaborate on the brief definitions given at the beginning of this paper.

1 Positive and Negative Moments. Fig. 42 will help the reader to visualize the following steps:

(a) Consider that all members intersecting at joint B have been cut off an infinitesimal distance away from the joint. (The distance is of course exaggerated in the sketch.)

(b) Let the reader imagine that he is standing on joint B and is keeping member BA in equilibrium by exerting a moment couple on the end of the cut member. Obviously he must exert compression on the side where the beam fibers are in compression and tension on the opposite side. In this case, he must exert a clockwise moment couple on the end of BA. If he now turns toward joint C, he must exert a counter-clockwise moment couple on it. From the definition:

If he exerts a clockwise moment couple on the end of the member, the moment is positive.

If he exerts a counter-clockwise moment couple on the end of the member, the moment is negative.

Hence, the signs of the moments at A, B, and C are as shown in the sketch.

2 Joint Rotation. Joint rotation angles are not used in the moment-distribution method proper. However, the slope-deflection equations are used in the derivation of several formulas in this paper, so the sign convention for joint rotation is given to help the reader in following the steps of the derivations.

Referring to Fig. 42, when the load was applied to beam BA, the effect was to apply a counter-clockwise moment on the end of beam BC, which caused the tangent to the elastic curve of this beam to rotate counter-clockwise also. By definition, this rotation was negative.

The moment at BA is positive under these conditions, because it is resisting the load. If the load were removed, the beam would deflect upward and the positive moment M_{ba} would rotate the

tangent to BA in a positive clockwise direction.

DISCUSSION OF SIGN CONVENTION

The sign convention which Professor Cross used in his paper was one with which structural engineers are ordinarily most familiar—i.e., a positive moment ca uses compression and a negative moment causes tension in the top fibers of a beam. For columns he suggested turning the drawing clockwise 90 deg and treating them as if they were beams.

The advantages of this system in the analysis of ordinary rectangular building frames are obvious. The disadvantages are more apparent when one tries to use it on a frame where members meet at various angles. The checking of a joint to see that the moments are in equilibrium is somewhat difficult, especially if there happen to be more than four members meeting at the particular joint. Care must also be used in determining the unbalanced moment at a joint where more than two members meet.

Hence, it has seemed to the author that the Wilson sign conventions[22] adopted for this paper are in general more satisfactory and less likely to lead to confusion, in that they automatically care for the sign of horizontal, vertical, and diagonal members. The checking of joints for equilibrium and the determination of unbalanced moments are facilitated, since the moments at a joint can be added algebraically. If the algebraic sum is zero, the joint is in equilibrium; if the sum is not zero, there is an unbalanced moment. A further although perhaps less important advantage of using the sign conventions selected for this paper is that multiplying by a carry-over factor of $+1/2$ instead of $-1/2$ allows less chance for error in computations.

In their text, "Statically Indeterminate Stresses," Parcel and Maney use sign conventions which are the reverse of those used by Messrs. Wilson, Richart, and Weiss. The choice between these two is largely a matter of personal preference.

It should be noticed that it is more satisfactory to transform both the Wilson and the Parcel and Maney sign conventions to the convention used by Hardy Cross before computing bending and shearing stresses. This transformation will be greatly facilitated if a rough sketch is made for each member showing the moments and forces acting on it.

Appendix B

FIXED-END MOMENTS, CARRY-OVER AND STIFFNESS FACTORS

Fixed-End Moment for Member Having Constant I: Fixed-end moment was defined at the beginning of this paper as the

[21] See discussion by Elmer F. Bruhn, Trans. A.S.C.E., vol. 96, 1932, p. 22.

[22] See footnote 6.

moment which would exist at the ends of a loaded member if those ends were held rigidly against rotation. These fixed-end moments for a beam of constant I can be found for different types of loadings in numerous texts or handbooks,[23] since they are also used in the slope-deflection equations. They can also be found quite readily by applying the three-moment equation and solving for the two unknown end moments or they can be determined by use of the M/EI diagram.

The latter method is applied to a uniformly loaded beam such as that shown in Fig. 3 as follows:

Area of the M/EI diagram for uniform loading is $WL^2/12EI$ (the area under a parabola of base L and maximum ordinate $WL/8EI$). Then the slope of the tangent to either end of the curved beam is numerically equal to the shear at the end when the span is loaded with the M/EI diagram. That is, $\theta_a = -\theta_b = WL^2/24EI$. To fix the ends in a horizontal position it is now necessary to apply a counter-clockwise moment at A and a clockwise moment at B, such that the ends will be rotated through an angle $-\theta_a = +\theta_b$. The M/EI diagram, when only the end moments act on the beam, will be a rectangle of height $M_a = M_b$ and length L. Hence,

$$\theta_a = \frac{-M_a L}{2} = \frac{WL^2}{24EI}$$

and

$$M_a = \frac{-WL}{12EI}$$

also

$$M_b = \frac{+WL}{12EI}$$

---

[23] References are "Structural Theory," by Sutherland and Bowman, p. 191, and "Stresses in Framed Structures," by Hool and Kinney, p. 485. (See also Fig. 41.)

The fixed-end moments can be worked out for other loadings in the same way.

Carry-Over Factor for Member Having Constant I: As previously stated under "Definitions," the carry-over factor for a beam fixed at A and simply supported at B is the ratio M_{ab}/M_{ba} and is equal to $+\frac{1}{2}$ for straight beams of constant section. This relation can be found from the three-moment equation, by use of the M/EI diagram or from the slope-deflection equations. By the latter method:

$$M_{ab} = 2EK(2\theta_a + \theta_b - 3R)$$

$$M_{ba} = 2EK(2\theta_b + \theta_a - 3R)$$

If the fixity is at end A, $\theta_a = 0$. $R = 0$, since there is no movement of supports. Whence $M_{ab}/M_{ba} = +\frac{1}{2}$.

Fixed-End Moments and Carry-Over Factors for Members Having Varying Moments of Inertia: A structure made up of members in which the moment of inertia varies from point to point along the span is quite complicated to analyze by any of the methods proposed so far. The Hardy Cross method can be advantageously applied to this type of frame, although considerable labor is involved in obtaining the required fixed-end moments, stiffness factors, and carry-over factors. In the general case, where the member not only has a changing cross-section, but also is unsymmetrical with respect to loading and form, it is necessary to find a fixed-end moment, stiffness factor, and carry-over factor to use with each end of the member, a total of six constants for the beam.

No attempt will be made in this paper to present the method of determining these constants. The reader is referred to a very able presentation offered in discussion of the Hardy Cross paper by A. W. Earl (Trans. Am.Soc.C.E., Vol. 96, 1933, p. 112). A somewhat different method was proposed by Prof. George E. Large in discussing the same paper (Trans. Am.Soc.C.E., Vol. 96, 1933, p. 101).